Exam Calculus 2

8 April 2019, 18:30-21:30

university of groningen

The exam consists of 6 problems. You have 180 minutes to answer the questions. You can achieve 100 points which includes a bonus of 10 points.

1. [**5+5+5=15 Points**] Let the function $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^4+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases} .$$

   (a) Is f continuous at $(x,y) = (0,0)$? Justify your answer.

   (b) Use the definition of directional derivatives to determine for which unit vectors $\boldsymbol{u} = (v,w) \in \mathbb{R}^2$ the directional derivative $D_{\boldsymbol{u}} f(0,0)$ exists.

   (c) Is f differentiable at $(x,y) = (0,0)$? Justify your answer.

2. [**15 Points**] Suppose $z = f(x,y)$ has continuous partial derivatives. Let us denote the function obtained by substituting $x = \mathrm{e}^r \cos\theta$ and $y = \mathrm{e}^r \sin\theta$ as $\tilde{z}$. Show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \mathrm{e}^{-2r}\left[\left(\frac{\partial \tilde{z}}{\partial r}\right)^2 + \left(\frac{\partial \tilde{z}}{\partial \theta}\right)^2\right].$$

3. [**6+3+6=15 Points**] Consider the curve parametrized by $\mathbf{r} : [0,1] \to \mathbb{R}^3$ with

$$\mathbf{r}(t) = t\,\mathbf{i} + \frac{\sqrt{2}}{2}t^2\,\mathbf{j} + \frac{1}{3}t^3\,\mathbf{k}.$$

   (a) Determine the length of the curve.

   (b) For each point on the curve, determine a unit tangent vector.

   (c) At each point on the curve, determine the curvature of the curve.

4. [**3+6+6=15 Points**] Let S be the ellipsoid in $\mathbb{R}^3$ defined by

$$x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 3$$

   which contains the point $(x_0, y_0, z_0) = (1, 2, 3)$.

   (a) Compute the tangent plane of S at the point (x_0, y_0, z_0).

   (b) Show that near the point (x_0, y_0, z_0) the ellipsoid S is locally given as the graph of a function over the (x,y) plane, i.e. there is a function $f : (x,y) \mapsto f(x,y)$ such that near (x_0, y_0, z_0) the ellipsoid is locally given by $z = f(x,y)$. Compute the partial derivatives f_x and f_y at (x_0, y_0) and show that the graph of the linearization of f at (x_0, y_0) agrees with the tangent plane found in part (a).

— please turn over —

(c) For a point $P = (x, y, z)$ in S, there is a box inscribed in S with corners (x, y, z), $(x, y, -z)$, $(x, -y, -z)$, $(x, -y, z)$, $(-x, y, z)$, $(-x, y, -z)$, $(-x, -y, -z)$ and $(-x, -y, z)$. Use the method of Lagrange multipliers to determine the box with largest possible volume.

5. [**4+5+6=15 Points**] Let a, b and c be continuous functions $\mathbb{R} \to \mathbb{R}$.

(a) Show that

$$\mathbf{F} = (a(x) + y + z)\,\mathbf{i} + (x + b(y) + z)\,\mathbf{j} + (x + y + c(z))\,\mathbf{k}$$

is conservative.

(b) Determine a scalar potential for $\mathbf{F}$.

(c) For $a(x) = x$, $b(y) = y^2$ and $c(z) = z^3$, compute the line integral along the straight line segment connecting the point $p = \mathbf{i}+\mathbf{j}$ to the point $q = \mathbf{j}+\mathbf{k}$. Verify this result using the potential function from part (b).

6. [**5+5+5=15 Points**] Let $f : \mathbb{R}^3 \to \mathbb{R}$, $(x, y, z) \mapsto f(x, y, z)$ be a function of class C^1, and let D be a solid region in $\mathbb{R}^3$. Let $\mathbf{n} = (n_1, n_2, n_3)$ be the outward normal unit normal vector to $S = \partial D$ (the boundary of D).

(a) If $\mathbf{a} \in \mathbb{R}^3$ is any constant vector and $\mathbf{F} = f\mathbf{a}$, show that $\nabla \cdot \mathbf{F} = \nabla f \cdot \mathbf{a}$.

(b) Use part (a) with $\mathbf{a} = \mathbf{i}$ to show that

$$\oint\!\!\!\oint_S f\, n_1 \,\mathrm{d}S = \iiint_D \frac{\partial f}{\partial x_1}\,\mathrm{d}V,$$

and obtain similar results by letting $\mathbf{a}$ equal $\mathbf{j}$ and $\mathbf{k}$.

(c) Define a *vector* quantity $\oint\!\!\oint_S f\,\mathrm{d}\mathbf{S} = \oint\!\!\oint_S f\mathbf{n}\,\mathrm{d}S$ by

$$\oint\!\!\!\oint_S f\mathbf{n}\,\mathrm{d}S = \left(\oint\!\!\!\oint_S fn_1\,\mathrm{d}S, \oint\!\!\!\oint_S fn_2\,\mathrm{d}S, \oint\!\!\!\oint_S fn_3\,\mathrm{d}S\right).$$

Show that with this notation

$$\oint\!\!\!\oint_S f\mathbf{n}\,\mathrm{d}S = \iiint_D \nabla f \mathrm{d}V\,,$$

where the right hand side is a vector whose components are obtained by integrating each of the scalar components of the integrand.

Solutions

1. (a) Substituting polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ for $(x,y) \neq (0,0)$ gives

$$\frac{x^2y^2}{x^4+y^4} = \frac{r^4\cos^2\theta\sin^2\theta}{r^4(\cos^4\theta+\sin^4\theta)} = \frac{\cos^2\theta\sin^2\theta}{\cos^4\theta+\sin^4\theta}.$$

For $r \to 0$, this has an limit that depends on θ. For example, for $\theta = \frac{\pi}{4}$, the limit is

$$\frac{\cos^2\theta\sin^2\theta}{\cos^4\theta+\sin^4\theta} = \frac{\frac{1}{2}\cdot\frac{1}{2}}{(\frac{1}{2})^2+(\frac{1}{2})^2} = \frac{1}{2}$$

which is not equal to $f(0,0) = 0$. The function f is hence not continuous at $(x,y) = (0,0)$.

(b) Let $\boldsymbol{u} = (v,w) \in \mathbb{R}^2$ be a unit vector, i.e. $v^2+w^2 = 1$. Then for the directional derivative $D_{\boldsymbol{u}}f(0,0)$ to exist the difference quotient

$$\frac{f(tv,tw)-f(0,0)}{t}$$

must have a limit for $t \to 0$. For $t \neq 0$, we have

$$\frac{f(tv,tw)-f(0,0)}{t} = \frac{\frac{t^4v^2w^2}{t^4(v^4+w^4)}-0}{t} = \frac{1}{t}\frac{v^2w^2}{v^4+w^4}.$$

This has a limit for $t \to 0$ only if $v^2w^2 = 0$, i.e. $v = 0$ or $w = 0$. This means that only directional derivatives in the direction of the x-axis and the y-axis exist. The derivatives are $f_x(0,0) = f_y(0,0) = 0$.

(c) f is not differentiable at $(x,y) = (0,0)$ because f is not continuous at $(x,y) = (0,0)$ as we have seen in part (a). Also if f was differentiable at $(x,y) = (0,0)$ then the directional derivative in part (b) would equal $\boldsymbol{u}\cdot\nabla f(0,0) = vf_x(0,0) + wwf_y(0,0) = 0$. But the directional derivative does not exist for every direction $\boldsymbol{u}$ as we have seen in part (b).

2. By the chain rule, we have

$$\begin{aligned}\frac{\partial\tilde{z}}{\partial r} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}\\ &= \frac{\partial z}{\partial x}e^r\cos\theta + \frac{\partial z}{\partial y}e^r\sin\theta\end{aligned}$$

and

$$\begin{aligned}\frac{\partial\tilde{z}}{\partial\theta} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial\theta}\\ &= \frac{\partial z}{\partial y}e^r\cos\theta - \frac{\partial z}{\partial x}e^r\sin\theta.\end{aligned}$$

So we have

$$\left[\frac{\partial\tilde{z}}{\partial r}\right]^2 = \left[\frac{\partial z}{\partial x}\right]^2(e^r\cos\theta)^2 + 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}e^r\cos\theta\cdot e^r\sin\theta + \left[\frac{\partial z}{\partial y}\right]^2(e^r\sin\theta)^2$$

and

$$\left[\frac{\partial \tilde{z}}{\partial \theta}\right]^2 = \left[\frac{\partial z}{\partial x}\right]^2 (e^r \sin\theta)^2 - 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} e^r \cos\theta \cdot e^r \sin\theta + \left[\frac{\partial z}{\partial y}\right]^2 (e^r \cos\theta)^2.$$

Adding the latter two equations yields

$$\begin{aligned}\left[\frac{\partial \tilde{z}}{\partial r}\right]^2 + \left[\frac{\partial \tilde{z}}{\partial \theta}\right]^2 &= (e^r)^2 \left[\frac{\partial z}{\partial x}\right]^2 (\cos^2\theta + \sin^2\theta) + (e^r)^2 \left[\frac{\partial z}{\partial y}\right]^2 (\sin^2\theta + \cos^2\theta) \\ &= e^{2r}\left[\left[\frac{\partial z}{\partial x}\right]^2 + \left[\frac{\partial z}{\partial y}\right]^2\right]\end{aligned}$$

which gives the desired equality

$$\left[\frac{\partial z}{\partial x}\right]^2 + \left[\frac{\partial z}{\partial y}\right]^2 = e^{-2r}\left[\left[\frac{\partial \tilde{z}}{\partial r}\right]^2 + \left[\frac{\partial \tilde{z}}{\partial \theta}\right]^2\right].$$

3. (a) The tangent vector

$$\mathbf{r}'(t) = 1\,\mathbf{i} + \sqrt{2}t\,\mathbf{j} + t^2\,\mathbf{k}.$$

has length

$$|\mathbf{r}'(t)| = \sqrt{1+2t^2+t^4} = 1+t^2\,.$$

The length of the curve is hence

$$\int_0^1 |\mathbf{r}'(t)|\mathrm{d}t = \int_0^1 (1+t^2)\mathrm{d}t = t + \frac{1}{3}t^3\Big|_{t=0}^{t=1} = 1 + \frac{1}{3} = \frac{4}{3}.$$

(b) Normalizing the tangent vector found in part (a) gives the unit tangent vector at the point $\mathbf{r}(t)$ as

$$\mathbf{T}(t) = \frac{1}{|\mathbf{r}'(t)|}\mathbf{r}'(t) = \frac{1}{1+t^2}(1\,\mathbf{i} + \sqrt{2}t\,\mathbf{j} + t^2\,\mathbf{k}).$$

(c) The curvature κ of the curve at the point $\mathbf{r}(t)$ is given by

$$\kappa = \frac{1}{|\mathbf{r}'(t)|}\left|\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{T}(t)\right| = \frac{1}{1+t^2}\frac{\sqrt{2}}{1+t^2} = \frac{\sqrt{2}}{(1+t^2)^2}.$$

4. (a) The ellipsoid S is given by the zero-level set of the function $F(x,y,z) = x^2 + \frac{y^2}{4} + \frac{z^2}{9} - 3$. We can hence find a normal vector of the tangent plane of S at (x_0, y_0, z_0) from $\nabla F(x_0,y_0,z_0) = 2x_0\mathbf{i} + \frac{1}{2}y_0\mathbf{j} + \frac{2}{9}z_0\mathbf{k} = 2\mathbf{i} + 1\mathbf{j} + \frac{2}{3}\mathbf{k}$. The tangent plane is hence given by $\nabla F(x_0,y_0,z_0)\cdot(x-x_0, y-y_0, z-z_0) = 0$, i.e.

$$2(x-1) + (y-2) + \frac{2}{3}(z-3) = 0$$

or equivalently,

$$6x + 3y + 2z = 18.$$

(b) Using the fact that S is given by the zero-level set of the function F defined in part (a) the local existence of the function f follows from the Implicit Function Theorem if we can show that $\frac{\partial F}{\partial z}(x_0, y_0, z_0) \neq 0$. The latter follows from $\frac{\partial F}{\partial z}(x_0, y_0, z_0) = \frac{2}{9}z_0 = \frac{2}{3}$. The Implicit Function Theorem gives

$$f_x(x_0, y_0) = -\frac{F_x(x_0, y_0, z_0)}{.F_z(x_0, y_0, z_0)} = -\frac{2x_0}{\frac{2}{9}z_0} = -\frac{2}{\frac{2}{3}} = -3$$

and

$$f_y(x_0, y_0) = -\frac{F_y(x_0, y_0, z_0)}{F_z(x_0, y_0, z_0)} = -\frac{\frac{1}{2}y_0}{\frac{2}{9}z_0} = -\frac{1}{\frac{2}{3}} = -\frac{3}{2}.$$

The linearization of f at (x_0, y_0) is given by

$$\begin{aligned} L(x, y) &= f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ &= 3 - 3(x - 1) - \frac{3}{2}(y - 2) \\ &= 9 - 3x - \frac{3}{2}y. \end{aligned}$$

The graph of the linearization is given by the equation $z = L(x, y)$ which agrees with the tangent plane found in part (a).

(c) The volume of the box is given by $V(x, y, z) = 8xyz$. It follows from the Theorem on Lagrange Multipliers that at a critical point $(x, y, z) \in S$ of V restricted to S there is a $\lambda \in \mathbb{R}$ such that $\nabla V(x, y, z) = \lambda \nabla F(x, y, z)$ where we use that the constraint S is given by the zero-level set of the function F defined in part (a). In order to find the critical points we have to solve the set of equations

$$\begin{aligned} V_x(x, y, z) &= \lambda F_x(x, y, z), \\ V_y(x, y, z) &= \lambda F_y(x, y, z), \\ V_z(x, y, z) &= \lambda F_z(x, y, z), \\ F(x, y, z) &= 0. \end{aligned}$$

for x, y, z and λ. This gives

$$\begin{aligned} 8yz &= \lambda 2x, \\ 8xz &= \lambda \frac{1}{2}y, \\ 8xy &= \lambda \frac{2}{9}z, \\ x^2 + \frac{y^2}{4} + \frac{z^2}{9} &= 3, \end{aligned}$$

As for $x = 0$, $y = 0$ or $z = 0$, $V(x, y, z) = 0$ we can assume $x, y, z \neq 0$. We then get

$$\begin{aligned} 4\frac{yz}{x} &= \lambda, \\ 16\frac{xz}{y} &= \lambda, \\ 36\frac{xy}{z} &= \lambda, \\ x^2 + \frac{y^2}{4} + \frac{z^2}{9} &= 3. \end{aligned}$$

Equating the left sides of the first and the second equality gives $y = 2x$. Equating the left sides of the first and the third equality gives $z = 3x$. This allows one to eliminate y and z in the last equality to get $3x^2 = 3$ which gives $x = 1$ (or $x = -1$ which we can discard by symmetry). So we get $y = 2x = 2$ and $z = 3x = 3$. The largest volume is hence $V = 8 \cdot 1 \cdot 2 \cdot 3 = 48$.

5. (a) Since $\mathbf{F}$ is defined on a simply connected domain it is sufficient to show that $\nabla \times \mathbf{F} = 0$ in order to prove that $\mathbf{F}$ is conservative.:

$$\nabla \times \mathbf{F}(x, y, z) = (\partial_y F_z - \partial_z F_y, \partial_z F_x - \partial_x F_z, \partial_x F_y - \partial_y F_x) = (1-1, 1-1, 1-1) = 0.$$

(b) The potential function $f : \mathbb{R}^3 \to \mathbb{R}$ satisfies $\nabla f(x, y, z) = \mathbf{F}$, i.e.

$$\frac{\partial f}{\partial x} = a(x) + y + z\,, \quad (1)$$

$$\frac{\partial f}{\partial y} = x + b(y) + z\,, \quad (2)$$

$$\frac{\partial f}{\partial z} = x + y + c(z)\,. \quad (3)$$

From Eq. (**??**) we get $\frac{\partial f}{\partial x} = a(x) + y + z$. Integrating with respect to x gives $f(x, y, z) = A(x) + yx + zx + g(y, z)$, where A is an integral of a (note that different choices for A differ only by constants which can be absorbed in the function g). Differentiating this f with respect to y should agree with the right hand side of Eq. (**??**). Equating the two gives $\frac{\partial g(y,z)}{\partial y} = b(y) + z$. Integrating with respect to y gives $g(y, z) = B(y) + zy + h(z)$ where B is an integral of b (similarly to above different choices for B differ by constants which can be absorbed in the function h). Hence $f(x, y, z) = A(x) + yx + zx + B(y) + yz + h(z)$. Differentiating this f with respect to z should agree with the right hand side of Eq. (**??**). Equating the two gives $c(z) = h'(z)$. Integrating with respect to z gives $h(z) = C(z) + d$ where $d \in \mathbb{R}$ is a constant. Hence

$$f(x, y, z) = A(x) + B(y) + C(z) + xy + xz + yz + d\,.$$

(c) Here $\mathbf{F}(x, y, z) = (x + y + z, x + y^2 + z, x + y + z^3)$. As a parametrization of the straight line segment connecting $\mathbf{p}$ and $\mathbf{q}$ we choose $\mathbf{r} : [0, 1] \to \mathbb{R}^3$, $t \mapsto (1 - t)\mathbf{p} + t\mathbf{q} = (1 - t, 1, t)$. Hence

$$\begin{aligned}\int_{\mathbf{pq}} \mathbf{F} \cdot \mathrm{d}\mathbf{s} &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t = \int_0^1 (2, 2, 2 - t + t^3) \cdot (-1, 0, 1)\,\mathrm{d}t \\ &= \int_0^1 (t^3 - t)\,\mathrm{d}t = [\frac{1}{4}t^4 - \frac{1}{2}t^2]_0^1 = -\frac{1}{4}\,.\end{aligned}$$

Following part (a) the potential function is $f(x, y, z) = \frac{1}{2}x^2 + \frac{1}{3}y^3 + \frac{1}{4}z^4 + xy + xz + yz + d$. Hence $f(q) - f(p) = \frac{1}{3} + \frac{1}{4} - \left(\frac{1}{2} + \frac{1}{3}\right) = -\frac{1}{4}$.

6. a) $a \in \mathbb{R}^3$, $F = fa = fa_1 i + fa_2 j + fa_3 k$

$$\Rightarrow \nabla \cdot F = \frac{\partial}{\partial x}(fa_1) + \frac{\partial}{\partial y}(fa_2) + \frac{\partial}{\partial z}(fa_3)$$

$$= a_1 f_x + a_2 f_y + a_3 f_z = a \cdot \nabla f = \nabla f \cdot a$$

b) let $a = i$, $F = fa = fi$

$$\Rightarrow \nabla \cdot F = f_x$$

also $F \cdot n = fi \cdot n = f n_1$

By Gauß' Divergence Theorem

$$\underset{S}{\oiint} \vec{F} \cdot d\vec{S} = \underset{D}{\iiint} \nabla \cdot \vec{F} \, dV$$

where we use '$\rightarrow$' to indicate vectors...

$$\| \quad \underset{S}{\oiint} \vec{F} \cdot \vec{n} \, dS$$

Plugging in $\vec{F} \cdot \vec{n} = f n_1$ and $\nabla \cdot \vec{F} = f_x$ gives the result.

Similarly for $a = j$ and $a = k$

c) just plug in the components...